Constructing the Real Numbers via Dedekind Cuts

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1 Dedekind Cuts

There are many ways to construct the set of real numbers $\mathbb{R}$, but one particularly simple method is using Dedekind cuts. In order to prove a more intuitive way to understand real numbers, Richard Dedekind defined the numbers in $\mathbb{R}$ by the way in which they split the set of rational numbers into disjoint sets. A number $r \in \mathbb{R}$ can split $\mathbb{Q}$ into two sets: $L : (\frac{a}{b}) < r$ and $(\frac{a}{b}) > r$. Thus, a real number is simply defined as the set of rational numbers less than itself; aka, L. Here, we know that L is non-empty, bounded above (by r) and has no maximum element, since the rationals are infinite. Furthermore, if $\frac{a}{b} \in L$ and $0 < \frac{c}{d} < \frac{a}{b}$ then $\frac{c}{d} \in L$.

2 Addition

We can define the additional of two real numbers L_1 and L_2 as the element-wise summation of the two sets.

Exercise: Demonstrate that if L_1 and L_2 are lower cuts, then so is $L_1 + L_2$.

Assume that addition of the real numbers is, as defined above, the element-wise summation of the two sets representing the two real numbers being added. Let p be the real number represented by the set L_1, and likewise for some q and L_2. Then, if $r_1 = \frac{a}{b} \in L_1$ and $r_1 = \frac{c}{d} \in L_2$ then $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and since $\frac{a}{b} < p$ and $\frac{c}{d} < q$, we know that $\frac{ad+bc}{bd} < p + q$ for any $a, b, c, d \in \mathbb{Z}$. Therefore, $L_1 + L_2$ is a lower cut.

We can also demonstrate the typical properties of addition that make the set of real numbers a field. One example is shown below:

Exercise: Demonstrate that $L_1 + L_2 = L_2 + L_2$.

$L_1 + L_2$ is represented by the element-wise addition of the two sets. We know that addition on the set of real numbers is commutative. Therefore, for any element in $L_1 + L_2$, the corresponding element of $L_2 + L_1$ is the same. Since the sets have the same elements, this means that $L_1 + L_2 = L_1 + L_2$.

We can also extend the same argument to demonstrate other properties of addition, such as commutativity.

3 Multiplication

Multiplication of two real numbers can be defined similarly; that is, $L_1 * L_2$ is the element-wise product of each element in the two sets.

Exercise: Demonstrate that if L_1 and L_2 are lower cuts, then so is $L_1 * L_2$.

Assume that multiplication of the real numbers is, as defined above, the element-wise product of the two sets representing the two real numbers being multiplied. Let p be the real number represented by the set L_1, and likewise for some q and L_2. Then, if $r_1 = \frac{a}{b} \in L_1$ and $r_1 = \frac{c}{d} \in L_2$ then $\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$ and since $\frac{a}{b} < p$ and $\frac{c}{d} < q$, we know that $\frac{ac}{bd} < p * q$ for any $a, b, c, d \in \mathbb{Z}$. Therefore, $L_1 * L_2$ is a lower cut.

We can also demonstrate the typical properties of multiplication that make the set of real numbers a field. One example is shown below:

Exercise: Demonstrate that $L_1 * (L_2 + L_3) = L_1 * L_2 + L_1 * L_3$.

The real number addition and multiplication is performed via element-wise operations on the rational numbers contained within the corresponding sets. For elements $\frac{a}{b} \in L_1, \frac{c}{d} \in L_2, \frac{e}{f} \in L_1$ we observe that $\frac{a}{b} * (\frac{c}{d} + \frac{e}{f}) = \frac{acf+ade}{bdf}$ while $\frac{a}{b} * \frac{c}{d} + \frac{a}{b} * \frac{e}{f} = \frac{ac}{bd} + \frac{ae}{bf} = \frac{acbf+abde}{b^2df} = \frac{acf+ade}{bdf}$. This indicates $\frac{a}{b} * (\frac{c}{d} + \frac{e}{f}) = \frac{a}{b} * \frac{c}{d} + \frac{a}{b} * \frac{e}{f}$. Since this is true for all elements of L_1, L_2, L_3, $L_1 * (L_2 + L_3) = L_1 * L_2 + L_1 * L_3$.

4 Ordering

The order of real numbers can be defined through the subset operator. The real numbers represented by the sets L_1 and L_2 can be ordered like so: $L_1 \leq L_2$ if $L_1 \subseteq L_2$.

Exercise: Demonstrate that if L_1 and L_2 are lower cuts, then either $L_1 \leq L_2$ or $L_2 \leq L_1$.

Let p be the real number represented by L_1, and likewise for q and L_2. If $L_1, L_2 \in \mathbb{R}$, then these sets are the sets of all real numbers such that $\frac{a}{b} < p$ and $\frac{c}{d} < q$. Thus, L_1 is the set of all elements $\frac{a}{b} < p$ and L_2 is the set of all elements $\frac{c}{d} < q$. If $p = q$, then $L_1 = L_2$; the sets have the same elements. Otherwise, the sets will have unequal numbers of elements In this case, either $L_1 \subset L_2$ or $L_2 \subset L_1$. Without loss of generality, if $L_1 \subset L_2$, then $L_1 \cap L_2 = \{\frac{c}{d} \in L_2 : \frac{c}{d} > p\}$. Since L_2 contains elements greater than p, $q > p$ and $L_2 > L_1$.

5 Completeness

A subset $S \subseteq \mathbb{R}^+$ is bounded above if $\exists L_1$ such that $L \leq L_1 \forall L \in S$, making L_1 an upper bound for S. The upper bound that is less than all other upper bounds for S is known as the supremum, or $\sup S$.

Exercise: Prove that if $S \subseteq \mathbb{R}^+$ has a least upper bound then the least upper bound is unique.

Proceed with a proof by contradiction: Assume that S has two unique least upper bounds, $\sup S_1$ and $\sup S_2$. Then, $\sup S_1 \neq \sup S_2$ since if they are equal, this contradicts the initial assumption of uniqueness. If $\sup S_1 \neq \sup S_2$ then $\sup S_1 > \sup S_2$ or $\sup S_1 < \sup S_2$. Both $\sup S_1 > \sup S_2$ and $\sup S_1 < \sup S_2$ violate the definition that $\sup S$ is the least upper bound of S, since there is an upper bound that is less than the assumed supremum, making either $\sup S_1$ or $\sup S_2$ not the least upper bound. Therefore, the least upper bound must be unique.

Exercise: Prove that if $S \subseteq \mathbb{R}^+$ is bounded then $\sup S$ exists.

Proceed with a proof by contradiction: assume that S is bounded but there is no $\sup S$. If $\sup S$ does not exist then either S is unbounded (violating the initial assumption) or the set of all upper bounds of S has no minimum. Even if the latter is the case, then there is no upper bound to S. This is because, for any feasible bound that one might take, there is always a bound that is smaller than that, since $\sup S$ does not exist and there is no least upper bound. This requires S to not have an upper bound, which contradicts the initial assumption.

6 Computability

By one definition, a real number is considered "computable" if we can define a Boolean function B that meets several criteria. First, $\exists r \in \mathbb{Q}^+$ for which $B(r) = 1$ and another such $\exists r \in \mathbb{Q}^+$ for which $B(r) = 0$. Furthermore, if $B(r) = 1$, then $\exists s > r$ with $B(s) = 1$ and if $B(r) = 1$ and $B(s) = 0$ then $r < s$. If these criteria are met then B is a lower cut function for r

Exercise: Show that if $B : \mathbb{Q}^+ \to \{0, 1\}$ is a lower cut function, then
$L := \{r \in \mathbb{Q}^+ : B(r) = 1\}$ is a lower cut.

1. If B is a lower cut function, then there exists $r \in \mathbb{Q}^+$ such that $B(r) = 1$. Therefore, it is known that L must be non-empty.
2. Furthermore, if $B(r) = 1$ and $B(s) = 0$ then $r < s$. Because the rational numbers are ordered, for rational numbers a, b, c, if $a < b$ and $b < c$ then $a < c$. Therefore, all values mapped to 0 by B must be greater than all values mapped to 1 by B. Thus, B maps all values r less than some real number to a value of 1. In more mathematical terms, $\forall r \in L, \exists s \notin L > r$. This means that L is bounded above, and that if $r_1 \in L$ and $r_2 < r_1$, $r_2 \in L$.

3. Finally, we know that $\forall r$ such that $B(r) = 1$, $\exists s > r$ with $B(s) = 1$. This indicates that L has no maximum element, since if it did, there would not exist $s > r$ with $B(s) = 1$ for the maximum element.

   Therefore, L has all of the criteria of a lower cut.