MTH311 Final Paper

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Contents

1 Introduction

This document is a compilation of my work this fall semester in MTH311: Advanced Calculus at University of Massachusetts Dartmouth. It covers my work in disseminating and becoming comfortable with the foundation and basics of real analysis, including construction of the sets of natural, rational, and real numbers, limits, a discussion of irrationality, continuity, Cauchy sequences, and sequences of functions. Each section typically contains a brief summary of the material being studied, or an explanation of the axioms that I am using to solve the problem at hand, followed by a few proof-based exercises related to the topic. It is my hope that the proofs are easy to follow, serve to elucidate the problems at hand, and represent my understanding of the topic.

2 Zermelo-Fraenkel Set Theory and the Set of Natural Numbers

2.1 Background

Zermelo-Fraenkel Set Theory was created to avoid certain paradoxes that could occur when working with sets, such as Russell's Paradox - for sets that are not members of themselves, if a property determines a set A, then A is a member of itself if and only if A is not a member of itself. Thus, Zermelo and Fraenkel created axioms to legitimize set theory [1].

From the axioms, Zermelo-Fraenkel Set Theory allows for the expression of natural numbers from sets. The natural number 0 is expressed as the empty set $\emptyset$, 1 is expressed as $\{\emptyset\}$, 2 is expressed as $\emptyset \cup \{\emptyset\}$ or $\{\emptyset, \{\emptyset\}\}$, 3 is expressed as $\emptyset \cup \{\emptyset, \{\emptyset\}\}$ or $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$, and so forth. This follows from the axiom that $w^+ := w \cup \{w\}$, and it "reifies" the process of unlimited counting through the successor function, which for a natural number n is defined as n^+. The successor of a natural number is simply the union of the empty set with the natural number itself

2.2 Addition

The Peano Axioms further explore the properties of the successor function for the set of natural numbers $\mathbb{N}$.

1. There is no natural number n with $n^+ = 0$
2. For all natural numbers m, n, if $m^+ = n^+$ then $m = n$
3. If $S \subseteq \mathbb{N}$ has the properties $0 \in S$ and $ifn \in Sthenn^+inS$ then $S = \mathbb{N}$

Thus, addition of natural numbers is a function mapping two natural numbers to another, defined as follows:

1. $m + 0 = m$ for all $m \in \mathbb{N}$

2. $m + n^+ = (m + n)^+$ for all $m, n \in \mathbb{N}$

For example, the simplistic 3 + 2 would be defined as follows:

$$\begin{aligned} 3 + 2 = 3 + 1^+ &= (3 + 1)^+ \\ &= (3 + 0^+)^+ \\ &= ((3 + 0)^+)^+ \\ &= ((3)^+)^+ \end{aligned} \tag{1}$$

Exercise: Argue why $m + n$ is defined for all $m, n \in \mathbb{N}$.

We know that if $n \in \mathbb{N}$, then $n^+ \in \mathbb{N}$ from the Peano Axioms. Since $m + n$ is defined recursively, it will eventually reduce to a repeated number of successor operations. Since if n is defined, then n^+ must be defined, it must be true that $m + n$ is defined for all $m, n \in \mathbb{N}$.

Exercise: Prove that the binary addition operation is commutative.

We proceed with a proof by induction.

1. Base case: $m + 0 = 0 + m$
2. $m + 0 = m$ as stated by the definition of the addition function.
3. To prove that $0 + m = m$, proceed with another proof by induction.
   (a) Base case: $0 + 0 = 0$, as stated by the Peano Axioms
   (b) Assume $0 + k = k$ is true
   (c) It follows that $0 + k + 1 = 0 + k^+ = (0 + k)^+ = k^+$ by the definition of the addition function.
   (d) Therefore, $0 + m = m$.
4. Therefore, because $m + 0 = m$ and $0 + m = m$, then $m + 0 = 0 + m$
5. Then, assume $m + k = k + m$ is true and prove that $m + k^+ = k^+ + m$
6. Firstly, we demonstrate that $m + 1 = 1 + m$
   (a) Proceed with another proof by induction.
   (b) Base case: $0 + 1 = (0 + 0)^+ = 1 = 1 + 0$ as stated in the Peano Axioms
   (c) Assume $k + 1 = 1 + k$
   (d) It follows that $(k + 1)^+ = 1 + k^+$ by the definition of the addition function.
   (e) Therefore, $m + 1 = 1 + m$
7. Finally, assume $m + k = k + mistrue$
8. It follows that $m + k + 1 = k + m + 1 = k + 1 + m$

9. Therefore, $m + n = n + m$ for all $m, n \in \mathbb{N}$ by the Principle of Mathematical Induction.

Exercise: Prove that the binary addition operation is associative.

Proceed with a proof by induction. Since we know that addition is commutative, $m + n = n + m$ for all $m, n \in \mathbb{N}$.

1. Base case: Prove $(m + n) + 0 = m + (n + 0)$
   (a) $(m + n) + 0 = m + n$ by the definition of the addition function
   (b) $m + (n + 0) = (n + 0) + m = n + m = m + n$ by the definitions of addition and association
   (c) Therefore $(m + n) + 0 = m + (n + 0)$
2. Then, assume $(m + n) + k = m + (n + k)$
3. $(m + n) + (k + 1) = (m + n) + k^+ = ((m + n) + k)^+$ by the definition of addition
4. In addition, $((m+n)+k)^+ = (m+(n+k))^+$ by the definition of addition
5. $(m + (n + k))^+ = m + (n + k)^+ = m + (n + k^+) = m + (n + (k + 1))$ by the definition of addition
6. Therefore, $(m + n) + p = m + (n + p)$ by the Principle of Mathematical Induction.

Exercise: Prove the cancellation property that $m + p = n + p \implies m = n$

Proceed with a proof by induction.

1. Base Case: if $m + 0 = n + 0$ then $m = n$. $m + 0 = m$ and $n + 0 = n$ by the definition of the addition function, therefore $m = n$.
2. Assume that $m + k = n + k \implies m = n$.
3. Therefore, k can be canceled like so: $m+k+1 = n+k+1 \implies m+1 = n+1$
4. Furthermore, $m + 1 = m^+$ and $n + 1 = n^+$ so $m^+ = n^+$, meaning $m = n$ by the Peano Axioms.
5. Therefore $m+k = n+k \implies m = n \implies m+k+1 = n+k+1 \implies m = n$
6. Therefore, $m + p = n + p \implies m = n$ by the Principle of Mathematical Induction.

2.3 Multiplication

Multiplication on the set of natural numbers is also defined using the successor function as follows:

1. $m * 0 = 0$ for all $m \in \mathbb{N}$

2. $m * n^+ = (m * n) + m$ for all $m, n \in \mathbb{N}$

Exercise: Argue why $m * n$ is defined for all $m, n \in \mathbb{N}$.

We know that if $n \in \mathbb{N}$, then $n^+ \in \mathbb{N}$ from the Peano Axioms. Since $m * n$ is defined recursively, it will eventually reduce to a repeated number of successor operations. Since if n is defined, then n^+ must be defined, it must be true that $m * n$ is defined for all $m, n \in \mathbb{N}$.

To proceed with certain proofs for multiplication, it is important to prove that multiplication distributes over addition.

Lemma. Multiplication of natural numbers distributes over addition.
$(m + n) * p = (m * p) + (n * p)$

Proceed with a proof by induction.

1. Base Case: $(m + n) * 0 = 0$ and $(m * 0) + (n * 0) = 0 + 0 = 0$ therefore $(m + n) * 0 = (m * 0) + (n * 0)$
2. Assume that $(m + n) * k = (m + k) * (n + k)$
3. It follows that $(m+n)*(k+1) = ((m+n)*k)+(m+n) = ((m*k)+(n*k)) + m + n$ by the definition of multiplication and induction hypothesis
4. Furthermore, $((m*k)+(n*k))+m+n = ((m*k)+m)+((n*k)+n) = (m * k^+) + (n * k^+)$ by the commutativity and associativity of addition
5. Therefore $(m+n)*p = (m*p)+(n*p)$ by the Principle of Mathematical Induction

This allows for the proofs in the following exercises.

Exercise: Prove that $m * 1 = m$ and $1 * m = m$ for all $m \in \mathbb{N}$

Proceed with a proof by induction.

1. Base Case: $1 * 0 = 0$ by the definition of multiplication.
2. Assume $1 * k = k = k * 1$
3. Induction step: Prove that $1 * (k + 1) = k + 1 = (k + 1) * 1$
4. $1 * (k + 1) = 1 * k^+ = (1 * k) + 1$ by the definition of multiplication
5. $(1 * k) + 1 = k + 1$ by induction hypothesis
6. $(k + 1) * 1 = (1 * k) + (1 * 1) = k + 1$ by distributive property proven previously.
7. Thus $1 * (k + 1) = k + 1 = (k + 1) * 1$
8. Therefore, $m * 1 = m$ and $1 * m = m$ for all $m \in \mathbb{N}$ by the Principle of Mathematical Induction

Exercise: Prove that multiplication is commutative.

Proceed with a proof by induction.

1. Base Case: Prove that $m * 0 = 0 * m$ for all $m \in \mathbb{N}$ by proceeding with another proof by induction.
   (a) Base Case: $0 * 0 = 0$ by multiplication
   (b) Assume $0 * k = 0$
   (c) It follows that $0 * (k + 1) = 0 * k^+ = (0 * k) + 0 = 0 + 0 = 0$ by the definition of multiplication and induction hypothesis.
   (d) Therefore $m * 0 = 0 * m$ for all $m \in \mathbb{N}$
2. Assume $m * k = k * m$ for all $m \in \mathbb{N}$
3. It follows that $m * (k + 1) = (m * k) + m = m + (k * m)$ by the definitions of addition, multiplication, and the induction hypothesis.
4. $m + (k * m) = (k + 1) * m$ by the distributive property proven previously.
5. Therefore $m * n = n * m$ for all $m, n \in \mathbb{N}$ by the Principle of Mathematical Induction

Exercise: Prove that multiplication is associative.

Proceed with a proof by induction.

1. Base Case: $(m * n) * 0 = 0, m * (n * 0) = m * 0 = 0$, therefore $(m * n) * 0 = m * (n * 0)$
2. Assume $(m * n) * k = m * (n * k)$
3. It follows that $(m*n)*(k+1) = ((m*n)*k)+(m*n) = (m*(n*k))+(m*n)$ by the induction hypothesis and the distributive property.
4. $(m * (n * k)) + (m * n) = m * ((n * k) + n) = m * (n * (k + 1))$
5. Therefore, $(m*n)*p = m*(n*p)$ meaning that multiplication is associative by the Principle of Mathematical Induction.

Exercise: Prove the cancellation property $m * p = n * p \implies m = n$ for all $m, n, p \in \mathbb{N}$

Proceed with a proof by induction.

1. Base Case: let $p = 1$; $m * 1 = m$ and $n * 1 = n$ therefore $m * 1 = n * 1 \implies m = n$
2. Assume that $m * k = n * k \implies m = n$
3. Furthermore, assume $m * k^+ = n * k^+$
4. Then, $m * (k + 1) = (m * k) + m$. Since $n * (k + 1) = (n * k) + n$, we know $(m * k) + m = (n * k) + n$.

5. Since $(m * k) = (n * k)$ by the induction hypothesis, both sides can be cancelled by the cancellation property of addition. Therefore $m = n$
6. Thus $m * k^+ = n * k^+ \implies m = n$
7. Therefore $m * p = n * p \implies m = n$ for all $m, n, p \in \mathbb{N}$

2.4 From Natural Numbers to Integers

For further proofs, it is useful to extend the idea of natural numbers to integers, to include negative counting numbers as well as a positive. Establishing the integers is important for later investigations.

The integers can be represented as a group. A group is a set equipped with a binary operation - for example, the set of natural numbers with addition, $(\mathbb{N}, +)$. In a Grothendieck group, the operation is commutative and fulfulls other general properties. The set of integers with addition can be constructed as the Grothendieck group of $(\mathbb{N}, +)$. This group exists and represents the set of integers because if an inverse of the addition operation must be defined, then the result must be denoted as the inverse, in this case using the $-$ negative notation. In this group, the difference between natural numbers $n - m$ is defined. Then,

$$\forall n \in \mathbb{N} : \begin{cases} n := [n - 0] \\ -n := [0 - n] \end{cases} \tag{2}$$

This rule defines the set of integers $\mathbb{Z}$.

3 The Field of Rational Numbers

3.1 Definition and Equivalence

A rational number is a number that can be expressed as a fraction in canonical form. A fraction is an ordered pair of integers. The set of all integers $\mathbb{Z}$ is defined from the set of natural numbers, a task which will be completed later, so for now we focus on the set of positive rational numbers, which forms a field as both addition and multiplication can be defined.

First, equality of fractions is defined as such: $\frac{a}{b} \equiv \frac{c}{d}$ indicates $ad = bc$. Obviously, $\frac{a}{b} \equiv \frac{a}{b}$ as $ab = ab$, and if $\frac{a}{b} \equiv \frac{c}{d}$ then $\frac{c}{d} \equiv \frac{a}{b}$ because $ad = bc$ is the same as $ad = bc$ for all $a, b, c, d \in \mathbb{N}$ by the definition of $=$.

Furthermore, this relation is transitive. If $\frac{a}{b} \equiv \frac{c}{d}$ and $\frac{c}{d} \equiv \frac{e}{f}$, then $\frac{a}{b} \equiv \frac{e}{f}$. This is because $\frac{a}{b} \equiv \frac{c}{d} \implies ad = bc$ thus $c = \frac{ad}{b}$, and $\frac{c}{d} \equiv \frac{e}{f} \implies cf = de$ thus $c = \frac{de}{f}$. Thus, $\frac{de}{f} = \frac{ad}{b}$ and by the cancellation property of multiplication, $\frac{a}{b} = \frac{e}{f}$. This, combined with the above two statements, makes the binary relation $\equiv$ an equivalence relation.

In addition, we also know that in each equivalence class for $\equiv$ there is a reduced fraction $\frac{a}{b}$ such that a, b only have the common factor 1. This is because, if we assume $\frac{a}{b} \equiv \frac{c}{d}$ such that c and d both have a common factor, then $\frac{a}{b}$ can be obtained from $\frac{c}{d}$ by dividing both c and d by the GCD of c and d.

3.2 Addition

Addition of fractions is defined as $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

Exercise: Prove that addition of fractions respects the equivalence of fractions; that is, prove that $\frac{a}{b} \equiv \frac{p}{q}$ and $\frac{c}{d} \equiv \frac{r}{s} \implies \frac{a}{b} + \frac{c}{d} = \frac{p}{q} + \frac{r}{s}$

Direct Proof.

1. First, if $\frac{a}{b} \equiv \frac{p}{q}$ then $aq = bp$ thus $a = \frac{bp}{q}$ by the definition of fraction addition.
2. Second, if $\frac{c}{d} \equiv \frac{r}{s}$ then $cs = dr$ thus $c = \frac{dr}{s}$ by the definition of fraction addiion.
3. Thus $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} = \frac{\frac{bdp}{q} + \frac{bdr}{s}}{bd}$ which, through division and simplification of the fraction, equates to $\frac{p}{q} + \frac{r}{s}$
4. Therefore $\frac{a}{b} \equiv \frac{p}{q}$ and $\frac{c}{d} \equiv \frac{r}{s} \implies \frac{a}{b} + \frac{c}{d} = \frac{p}{q} + \frac{r}{s}$

3.3 Multiplication

Multiplication of fractions is defined as $\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$

Exercise: Prove that multiplication of fractions respects the equivalence of fractions; that is, prove that $\frac{a}{b} \equiv \frac{p}{q} and \frac{c}{d} \equiv \frac{r}{s} \implies \frac{a}{b} * \frac{c}{d} = \frac{p}{q} * \frac{r}{s}$

Direct Proof.

1. First, if $\frac{a}{b} \equiv \frac{p}{q}$ then $aq = bp$ thus $a = \frac{bp}{q}$ by the definition of fraction multiplication.
2. Second, if $\frac{c}{d} \equiv \frac{r}{s}$ then $cs = dr$ thus $c = \frac{dr}{s}$ by the definition of fraction multiplication.
3. Thus $\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd} = \frac{\frac{bp}{q} * \frac{dr}{s}}{bd} = \frac{\frac{bdpr}{qs}}{bd}$ which, through division and simplification of the fraction, equates to $\frac{pr}{qs}$
4. Furthermore, $\frac{p}{q} * \frac{r}{s} = \frac{pr}{qs}$ by the definition of fraction multiplication.
5. Therefore $\frac{a}{b} \equiv \frac{p}{q} and \frac{c}{d} \equiv \frac{r}{s} \implies \frac{a}{b} * \frac{c}{d} = \frac{p}{q} * \frac{r}{s}$

3.4 Order

Like the natural numbers, the rational numbers are also ordered. For all $\frac{a}{b}$ in canonical form, $\frac{a}{b} < \frac{c}{d}$ implies that $ad < bc$. There are several properties that make the set of rational numbers an ordered field. Many, such as commutativity and associativity of addition and multiplication operations, are similar to the natural numbers which were proven previously. Thus, four proof exercises will be presented here for new properties.

Exercise: Prove that for all $0 \neq r \in \mathbb{Q}$ there exists $r^* \in \mathbb{Q}$ such that $r * r^* = 1$

Proceed with direct proof.

1. if $r, r^* \in \mathbb{Q}$ then $r = \frac{a}{b}$ and $r^* = \frac{c}{d}$
2. Let $r^* = \frac{b}{a}$. Then $r * r^* = \frac{a}{b} * \frac{b}{a} = \frac{ab}{ab} = 1$

Exercise: Prove that for all $r \neq s \in \mathbb{Q}, r < sorr > s$

Proceed with a proof by contrapositive.

1. Assume $r \not< s$. Then $r > sorr = s$.
2. Then assume $r \not> s$. Then $r < sorr = s$
3. Since we know $r \not< s$ and $r \not> s$, $r = s$

Exercise: Prove that if $r < s$ then $r + t < s + t$ for $r, s, t \in \mathbb{Q}$.

Proceed via direct proof.

1. Let $r = \frac{a}{b}, s = \frac{c}{d}, t = \frac{e}{f}$
2. Assume $r < s$ thus $ad < bc$
3. $r + t = \frac{af+be}{bf}$ while $s + t = \frac{cf+de}{df}$
4. Therefore since $\frac{af+be}{bf} < \frac{cf+de}{df}$, then $(af + be)(df) < (cf + de)(bf)$
5. This simplies to $ad < bc$, which implies $r < s$

Exercise: Prove that for all $r, s \in \mathbb{Q}$, if $r > 0$ and $s > 0$ then $r * s > 0$.

Proceed with a direct proof.

1. If $r > 0$ then $r = \frac{a}{b}$ with $a, b > 0$
2. If $s > 0$ then $r = \frac{c}{d}$ with $c, d > 0$
3. $r * s = \frac{a}{b} * \frac{c}{d} = \frac{ab}{cd}$
4. For $\frac{ab}{cd} = 0$, $a = 0$ or $c = 0$ but we know this is not true.
5. Therefore for all $r, s \in \mathbb{Q}$, if $r > 0$ and $s > 0$ then $r * s > 0$

Archimedean ordering property: For every $r \in Q$ there is a natural number n for which $n > r$.

Proof: Proceed with a proof by cases.

1. Since r is a rational number, $r = \frac{a}{b}$ with $a, b \in \mathbb{N}$
2. Case 1: $b = 1$

(a) If $b = 1$, then $r = a$ and $r \in \mathbb{Z}$

(b) Since $b \in mathbbZ$, there exists $b + 1 \in N$ such that $b + 1 > b = r$

(c) Therefore, when $r = \frac{a}{b}$ and $b = 1$, there exists a natural number $n > r$

3. Case 2: $b \neq 1$

(a) On one hand, if $r = \frac{a}{b}$, $b > 1$, and $a > 0$ there exists a rational number $n = \frac{a}{1}$ with $a, b \in \mathbb{Z}$ meaning that $n > r$.

(b) On the other hand, if $r = \frac{a}{b}$, $b > 1$, and $a < 0$ there exists a rational number $n = \frac{a}{-1}$ with $a, b \in \mathbb{Z}$ meaning that $n > r$.

(c) Since in either case $n = \frac{a}{1}$, $n = a$ meaning that $n \in \mathbb{N}$

(d) Therefore, when $r = \frac{a}{b}$ and $b > 1$, there exists a natural number $n > r$

4. Since there exists a natural number $n > r$ for every rational number $r = \frac{a}{b}$ when $b = 1$, $b > 1$, and $b < 1$

4 Limits

Limits allow for exploration of long-term behaviors - for the rational numbers, it allows us to examine what happens when a sequence continues on infinitely. A sequence of rational numbers is a function that maps each natural number to a rational number; since $\mathbb{N}$ is countably infinite, the set of the terms of the sequence is also countably infinite.

For instance, for the sake of exercises, we can define the sequence $p(n)$ like so:

$p(1) = 1;$

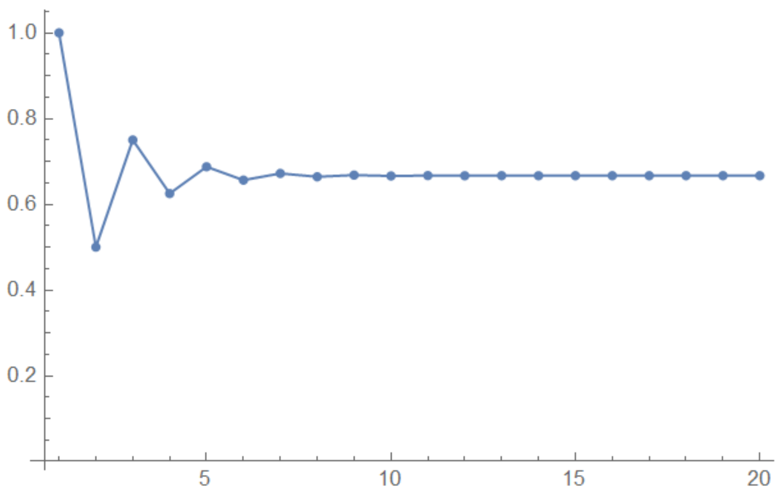
$p(n) = 1 - \frac{1}{2}p(n-1)$ for all $n \geq 2$;

Exercise: Prove that $p(n)$ is rational

Proceed with a proof by induction

1. Base Case: Since $p(1) = 1$, $p(1)$ is rational.
2. Assume $p(k) = 1 - \frac{1}{2}p(k-1)$ is rational
3. $p(k+1) = 1 - \frac{1}{2}p(k)$, and since $p(k)$ is rational by the induction hypothesis, $1 - \frac{1}{2}p(k)$ must be rational since two rational numbers multiplied or subtracted equal a rational number.
4. Therefore $p(n)$ is rational by the Principle of Mathematical Induction.

Exercise: Plot $p(n)$ versus n for $1 \leq n \leq 20$



Exercise: With $\epsilon(n) := p(n) - \frac{2}{3}$ find a recurrence relation for $\epsilon(n)$

$$\begin{aligned}\epsilon(n) &= p(n) - \frac{2}{3} \\ &= 1 - \frac{1}{2}p(n-1) - \frac{2}{3} \\ &= \frac{1}{3} - \frac{1}{2}p(n-1) \\ &= -\frac{1}{2}(p(n-1) - \frac{2}{3}) \\ &= -\frac{1}{2}\epsilon(n-1)\end{aligned}$$

Exercise: Show that $\epsilon(n) \to 0$ as $n \to \infty$

We can prove that $\epsilon(n) \to 0$ as $n \to \infty$ by showing that the absolute value of the function is non-increasing; at each step, ϵ gets close to 0.

Prove that $|\epsilon(n)| < |\epsilon(n-1)|$

1. By the recurrence relation, $|\epsilon(n)| = |-\frac{1}{2}\epsilon(n-1)|$
2. $\epsilon(1) = p(1) - \frac{2}{3} = \frac{1}{3}$
3. At each step of the sequence, the previous value is just multiplied by $\frac{1}{2}$. Therefore we can write $|\epsilon(n)| = (\frac{1}{3})(\frac{1}{2})^n$
4. We know that as n increases, $(\frac{1}{2})^n$ approaches 0, because $(\frac{1}{2})^{n+1} = \frac{1}{2}(\frac{1}{2})^n < \frac{1}{2}^n$.
5. Therefore $\epsilon(n) \to 0$ as $n \to \infty$

5 Irrationality

An irrational number is a number that cannot be written as a ratio $\frac{a}{b}$ with $a, b \in \mathbb{Z}$. A famous example, known to the Ancient Greeks, is $\sqrt{2}$, which the Pythagoreans argued was not commensurable with the unit 1 since there exist no whole numbers m and n such that $n * l = m * 1 = m$. There are many ways to prove that $\sqrt{2}$ is irrational; some involve contradiction, while others are constructive (and therefore more rigorous for those who do not believe that a statement can only be true or false).

5.1 Proving a number is irrational

Below we demonstrate that other numbers are irrational using similar techniques that are used to prove that $\sqrt{2}$ is irrational:

Exercise: Prove that $\sqrt{5}$ is irrational.

Proceed with a proof by contradiction.

1. Assume that for l where $l^2 = 5$, there exist $m, n \in \mathbb{N}$ such that $n * l = m * 1 = m$.
2. Let n be the smallest natural number for which nl is an integer.
3. Let $n* = n(l - 2)$. Then $n* = n(l - 2) = nl - 2n = m - 2n$ is an integer.
4. If $l \leq 2$ then $l^2 \leq 2^2 = 4$, however $l^2 = 5 > 4$, therefore $l > 2$
5. As such, $n* = n(l - 2) > 1$ so $n*$ is a natural number.
6. $n * l = n(l - 2)l = nl^2 - 2nl = 5n - 2m \in \mathbb{N}$.
7. $n*$ is a natural number such that $n * l$ is an integer.
8. However, if $l - 2 \geq 1$ then $l \geq 3sol^2 \geq 9 > 5$, therefore $l - 2 < 1$.
9. Therefore, $n* < n$ because $n* = n(l - 2) < n * 1$.
10. This forms a contradiction as we already assumed n is the smallest natural number such that nl is an integer. Thus $\sqrt{5}$ must be irrational.

Note that this proof works for any irrational square root, if in Step 3 you let $n* = n(l - k)$ where k^2 is the largest square number less than the number whose square is being proven irrational, and adjust the rest of the proof accordingly. No contradiction is formed when l is rational, because then one would find in Step 9 that $n* = n$ as $n* = n(l - k) = n * 1$, so there is no contradiction.

Exercise: Prove that $2^{1/3}$ is irrational.

Proceed with a proof by contradiction.

1. Assume $l = \frac{a}{b}$ such that $a, b \in \mathbb{N}$ and $GCD(a, b) = 1$.

2. Then $2 = l^3 = \frac{a^3}{b^3}$ so $a^3 = 2b^3$. Since a^3 is even, a must be even, s $a = 2m$ for some $m \in \mathbb{N}$.
3. Then $2b^3 = a^3 = 8m^3$ so $b^3 = 4m^3$. Thus b^3 is even and therefore b is even and $b = 2n$ for some $n \in \mathbb{N}$.
4. However, if both a and b are even, then $GCD(a, b) >= 2$, contradicting the original assumption that $GCD(a, b) = 1$

Exercise: Show that $\log_{10} 5$ is not a rational number (Challenge: Using a constructive proof)

Proceed with a proof by cases. Let $l = \log_{10} 5$ Case 1: $l + \frac{a}{b} > 2$

1. $\frac{a}{b} - l > 2 - 2l$ through algebraic manipulation
2. $\frac{a}{b} - l > 2 - 2l > 0$ since $l < 1$ as $\log_{10} 5 < log_{10}10 = 1$
3. Thus, $l \neq \frac{a}{b}$ when $l + \frac{a}{b} > 2$

Case 2: $l + \frac{a}{b} \leq 2$

1. $10^{\frac{a}{n}+l} < 10^2 = 100$
2. Therefore $10^{\frac{a}{n}-l} < \frac{100}{10^{2l}}$ through logarithm properties
3. This implies that $10^{\frac{a}{n}-l} < 4$ through logarithm properties.
4. Hence $\frac{a}{b} - l > 0$ since $10^0 = 1$.
5. Thus $2l < \frac{a}{b} + l < 2$ as $\frac{a}{b} + l \neq 2$ since $l < 1$
6. Therefore $l \neq \frac{a}{b}$ as $l < \frac{a}{b}$

5.2 The Babylonian Method for Approximating $\sqrt{2}$

Thousands of years ago, the Babylonians invented an iterative method for approximating the value of $\sqrt{2}$ that gets closer and closer to the actual value the more iterations are performed. Though it never reaches the exact value, since $\sqrt{2}$ is irrational, it is an example of a series whose limit approaches an irrational number. The method is outlined below:

$$x_1 = \frac{3}{3}$$
$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \forall n \geq 1$$

Exercise: Prove that $|x_n^2 - 2|$ approaches 0 as n increases

We do this by proving that $|x_{n+1}^2 - 2| < |x_n^2 - 2|$ for all $n \geq 1$. This is analogous to proving that the error is monotonically decreasing to 0. Proceed with a proof by induction.

1. Base Case: $x_n = \frac{3}{2}, x_{n+1} = \frac{\frac{3}{2}}{2} + \frac{1}{\frac{3}{2}} = \frac{17}{12} < \frac{18}{12} = \frac{3}{2}$ therefore $x_2 < x_1$
2. Induction Hypothesis: Assume $|x_k^2 - 2| < |x_{k-1}^2 - 2|$
3. Induction Step: Prove that $|x_{k+1}^2 - 2| < |x_k^2 - 2|$
   (a) By substituting in $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ and solving algebraically, we find that $|x_{k+1}^2 - 2| = \frac{(x_k^2-2)^2}{4x_k^2}$
   (b) Then, we prove that $x_k^2 > 2$ for all $k \geq 1$ by using another proof by induction
      i. Base Case: $x_1^2 = (\frac{3}{2})^2 = \frac{9}{4} > \frac{8}{4} = 2$
      ii. Induction Hypothesis: Assume $x_k^2 > 2$
      iii. Induction Step: Prove that $x_{k+1}^2 > 2$
         A. $x_{k+1}^2 = \frac{x_k^2}{4} + \frac{1}{x_k^2} + 1$ by algebraic manipulation.
         B. $\frac{x_k^2}{4} + \frac{1}{x_k^2} + 1 > 2$ if and only if $x_k^4 > 4 - 4x_k^2$ by algebraic manipulation
         C. Let $z = x_k^2$. It is known that z^2 increases more rapidly than z, and that $2^2 = 4 * 2 - 4 = 4$ is true. Since $z^2 > 4 - z$ for all $z > 2$, then since $x_k^2 > 2$, it is true that $x_k^4 > 4 - 4x_k^2$.
         D. Thus it follows that $\frac{x_k^2}{4} + \frac{1}{k_n^2} + 1 > 2$ therefore $x_{k+1}^2 > 2$
      iv. Because $x_1^2 > 2$ and $x_k^2 > 2 \implies x_{k+1}^2 > 2$, $x_k^2 > 2$ for all $n \geq 1$ by the Principle of Mathematical Induction
   (c) Since $x_k^2 > 2$, it follows that $|x_{k+1}^2 - 2| = \frac{(x_k^2-2)^2}{4x_k^2} < \frac{(x_k^2-2)^2}{8} < \frac{(x_{k-1}^2-2)^2}{8} = |x_k^2 - 2|$
   (d) As such, $|x_k^2 - 2| < |x_{k-1}^2 - 2| \implies |x_{k+1}^2 - 2| < |x_k^2 - 2|$
4. Therefore, $|x_{n+1}^2 - 2| < |x_n^2 - 2|$ for all $n \geq 1$ by the Principle of Mathematical Induction.

The square-root of 2 can also be thought of as an "infinitely long fraction" or a "continued fraction." If we write $x = \sqrt{2} + 1$, we can algebraically manipulate x to get $2 + \frac{1}{x}$, and keep substituting x with no end. Approximating this fraction allows us to obtain a sequence of rational numbers that convergences to $\sqrt{2}$.

Since in class we have been discussing the concept of mathematics as the communication of "ideas," and how all proofs should start as concepts, I have attempted to write the proofs in this section concisely, boiling down the proof to its main idea(s), rather than writing extremely detailed steps for solving the problem at hand.

Exercise: Prove that if the nth convergent of the infinite fraction is $\frac{p_n}{q_n}$ then the next convergent is $\frac{p_{n+1}}{q_{n+1}} = \frac{p_n+2q_n}{p_n+q_n}$

$\frac{p_{n+1}}{q_{n+1}}$ approximates $\sqrt{2}$, while the infinite fraction approximates $\sqrt{2}+1$. If we let each iterate of the infinite fraction be represented as S_n, then we know $S_n = \frac{p_n}{q_n} + 1$ and $S_{n+1} = \frac{p_{n+1}}{q_{n+1}} + 1$.

$S_{n+1} = 2 + \frac{1}{S_n}$ therefore $\frac{p_{n+1}}{q_{n+1}} + 1 = 2 + \frac{1}{\frac{p_n}{q_n}+1}$. Through algebraic manipulation, we find that $\frac{p_{n+1}}{q_{n+1}} = 1 + \frac{1}{\frac{p_n+q_n}{q_n}} = \frac{2q_n+p_n}{p_n+q_n}$.

Therefore, if the nth convergent of the infinite fraction is $\frac{p_n}{q_n}$ then the next convergent is $\frac{p_{n+1}}{q_{n+1}} = \frac{p_n+2q_n}{p_n+q_n}$

Exercise: Prove that if $\frac{p}{q}$ is rational then $\frac{p+2q}{p+q}$ is rational.

Let $p = \frac{a}{b}$ with $GCD(a,b) = 1$ and $q = \frac{c}{d}$ with $GCD(c,d) = 1$ and $a,b,c,d \in \mathbb{Z}$.

Then $\frac{p+2q}{p+q} = \frac{\frac{a}{b}+\frac{2c}{d}}{\frac{a}{b}+\frac{c}{d}} = \frac{\frac{ad+2bc}{bd}}{\frac{ad+bc}{bd}} = \frac{ad+2bc}{ad+bc}$ which is rational as $a,b,c,d \in \mathbb{Z}$.

Therefore if $\frac{p}{q}$ is rational then $\frac{p+2q}{p+q}$ is rational.

Exercise: Prove that if $a_n = \frac{p_{2n-1}}{q_{2n-1}}$ and $b_n = \frac{p_{2n}}{q_{2n}}$, then $a_n \leq a_{n+1} \leq b_n \leq b_{n+1}$

In general, for $a,b \in \mathbb{N}$, we know $\frac{a}{b} \leq \frac{a+2b}{a+b}$ when a,b both positive, because $2b > b$ for all $b \leq 1$.

Since $a_{n+1} = \frac{p_{2n}}{q_{2n}} = \frac{p_{2n-1}+2q_{2n-1}}{p_{2n-1}+q_{2n-1}} < \frac{p_{2n-1}}{q_{2n-1}} = a_n$. Furthermore, this argument can be extended without loss of generality to all other cases to prove $a_n \leq a_{n+1} \leq b_n \leq b_{n+1}$.

Exercise: Prove that $b_n - a_n \to 0$ as $n \to \infty$

Since b_n and a_n represent the even and odd iterates of the approximation, respectively, then we can write $b_n = a_n + \epsilon$, where ϵ represents the additional fraction added at the next iteration. We know that $\lim_{n\to\infty} \epsilon = 0$ from the nature of the continued fraction, therefore $b_n - a_n = 0$

Exercise: Prove that $a_n^2 < 2 < b_n^2$ as $n \to \infty$.

1. We know that $a_1 = 1$ and $b_1 = \frac{3}{2}$. Therefore $a_1^2 = 1 < 2$ and $b_1^2 = \frac{9}{4} > 2$.
2. Since a_n and b_n are rational numbers, and $b_n^2 = (1+\frac{1}{a_n})^2 = 1 + \frac{2}{a_n} + \frac{1}{a_n^2}$, we know that all $b_n^2 > 2$.
3. We also know from the previous exercises that $a_n \leq a_{n+1} \leq b_n \leq b_{n+1}$, hence a_n is monotonically increasing and b_n is monotonically decreasing.
4. From another previous exercise, we know that as n approaches infinity, $a_n = b_n$ hence $a_n^2 = b_n^2$. Therefore $a_n^2 < 2 < b_n^2$ as $n \to \infty$

6 Properties of Real Numbers from Lebl's Basic Analysis

6.1 Preliminaries

In this section, I stray from the material of the lectures and take an independent dive into some properties of real numbers using one of the resources recommended for this class - Lebl's Basic Analysis. This will help me better understand the numbers that form the foundation of real analysis.

It seems, to define the set of real numbers, it is important to understand certain properties of sets. From [2], an ordered set is a set such that for any pair of elements x, y, one of the following relations holds: $x < y, x = y, x > y$. These relations must also have the transitive property. From our previous work, we already know that $\mathbb{Q}$ and $\mathbb{Z}$ are ordered sets. Furthermore, these two sets are also ordered fields, as addition and multiplication are defined, $x < y \implies x + z < y + z$, and $x > 0, y > 0 \implies xy > 0$.

Furthermore, if S is an ordered set and $E \subset S$, then E is bounded with a bound of b if $b \in S$ such that $b \geq x$ (for upper bound) or $b \leq x$ (for lower bound) for all $x \in E$. It follows that the "least upper bound" is the bound b_0 that is the smallest such b that forms an upper bound of the set, and the "greatest lower bound" is the b that is the largest such b that forms a lower bound of the set. A set has the "least upper bound property" or "completeness property" if every nonempty subset $E \subset S$ that is bounded above has a least upper bound that is in S. From [2], there exists a unique ordered field $\mathbb{R}$ with the least-upper-bound property such that $\mathbb{Q} \in \mathbb{R}$. Since we know, from the least-upper-bound property and from our previous investigations, that $\mathbb{R}$ contains many other numbers besides the rationals, we can define $\mathbb{R} - \mathbb{Q}$ as the set of irrational numbers.

The reason that analysts use the set of real numbers is precisely because of its least-upper-bound property. $\mathbb{Q}$ does not have this property, which makes analysis impossible in many cases.

Analysts prove inequalities using the subsequent proposition from [2]: If $x \in \mathbb{R}$ such that $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x \leq 0$.

1. We know that if $x > 0$, then $0 < \frac{x}{2}$. This is because dividing a positive integer by a positive integer yields a positive rational number, and because dividing a rational number by a positive integer yields a smaller rational number.

2. If we let $\epsilon = \frac{x}{2}$, then $\epsilon < x$, which is a contradiction. Therefore $x \leq 0$.

6.2 The Archimedean Property

There are infinitely many rational numbers in any interval on the real number line. This comes from a theorem from [2]:

1. Archimedean property: If $x, y \in \mathbb{R}$ and $x > 0$ then $\exists n \in \mathbb{N}$ such that $nx > y$.
2. $\mathbb{Q}$ dense in $\mathbb{R}$: If $x, y \in \mathbb{R}$ ad $x < y$, then $\exists r \in \mathbb{Q}$ such that $x < r < y$.

But how can we prove these two facts? Here are my interpretations from [2]

Problem: Prove the Archimedean Property

For $nx > y$ to be true for all $x, y \in \mathbb{R}$, $\mathbb{N}$ must not be bounded. We already know that $\mathbb{N}$ is not bounded. This is because for any $n \in \mathbb{N}$, there exists $n^+ \in \mathbb{N}$ which is greater than n.

Suppose $\mathbb{N}$ were bounded by some least-upper-bound b; then if we let $m \in \mathbb{N}$ be a number in $b - 1 < m < b$, then $m + 1 > b$ and $m + 1 \in \mathbb{N}$ as well by the definition of the set of natural numbers, meaning that b is not the least-upper-bound which causes a contradiction. Therefore the Archimedean property must be true.

Problem: Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

To prove this, let us construct $r \in \mathbb{Q}$ such that $x < r < y$. Since $x \neq y$, we know that there must be some $n \in \mathbb{N}$ such that $n(y - x) > 1$ by the Archimedean Property, so $ny > 1 + nx$. We also know by the Archimedean Property that $nx < m$ for some rational number m.

Let m be the smallest element such that $nx < m$. Since $m - 1$ is smaller than m, this means that $m - 1 \leq nx$ (note $m \neq 0$ because $n \in \mathbb{N}$). Therefore $ny > 1 + nx \geq m$, so $y > \frac{m}{n}$ which is a rational number. Thus $x < \frac{m}{n} < y$ indicating that $\mathbb{Q}$ dense in $\mathbb{R}$.

7 Continuity

7.1 Proving Continuity of a Function

Continuity of a function is an important aspect of many proofs in mathematics - but how can we define "continuity" in a rigorous manner? To do this, we use an "delta-epsilon" argument or proof. To check that a function f is continuous, we examine whether there exists $\delta > 0$ such that $|x - x_0|$ implies that $|f(x) - f(x_0)| < \epsilon$ for every $\epsilon > 0$. For now, we will work with functions that are defined on the set of rational numbers, meaning that $\delta, \epsilon, x, x_0 \in \mathbb{Q}$ and the function f maps $\mathbb{Q} \to \mathbb{Q}$. Essentially, we show that $f(x)$ and $f(x_0)$ are close if x and x_0 are close enough.

Exercise: Prove that $f(x) = ab + b$ is continuous, given $a, b \in \mathbb{Q}$.

To test for continuity, we test that $\exists \delta \in \mathbb{Q}$ and $0 < \delta$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$ with $0 < \epsilon \in \mathbb{Q}$.

1. It is known that $|f(x) - f(x_0)| = |ax + b - ax_0 - b| = a|x - x_0|$.

2. If we let $\delta = \frac{\epsilon}{a}$, then $|x - x_0| < \delta \implies a|x - x_0| < \epsilon$ since $a|x - x_0| < a\delta = a\frac{\epsilon}{a} = \epsilon$ since a is a constant.
3. Therefore $ax + b$ is continuous at all $x \in \mathbb{Q}$

Exercise: Prove that $f(x) = x^2$ is continuous on the set of rational numbers.

We proceed with a delta-epsilon proof.

1. $|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0||x + x_0|$.
2. Furthermore, we can extract $|x - x_0|$ from $|x + x_0|$ through the following equality: $|x + x_0| = |x - x_0 + 2x_0|$
3. Let $\delta = min(\frac{\epsilon}{|1+2x_0|}, 1)$.
4. Then $|x - x_0 + 2x_0| \leq |x - x_0| + |2x_0| < 1 + 2|x_0|$.
5. Therefore $|x^2 - x_0^2| = |x - x_0||x + x_0| < (1 + 2|x_0|)|x - x_0| < (1 + 2|x_0|)\delta < \epsilon$
6. Thus $f(x) = x^2$ is continuous on the set of rational numbers.

A similar argument can be used to prove the continuity of other polynomials.

Exercise: Prove that $f(x) = x^2 + x$ is continuous on the set of rational numbers.

We proceed with a delta-epsilon proof.

1. $|f(x) - f(x_0)| = |x^2 + x - x_0^2 - x_0| = |x - x_0||x + x_0 + 1|$
2. Let $x_0 \in \mathbb{Q}$ and for $\epsilon > 0 \in \mathbb{Q}$ let $\delta = min(\frac{\epsilon}{1+|2x_0+1|}, 1)$.
3. $|x + x_0 + 1| = |x - x_0 + 2x_0 + 1| = |x - x_0| + |2x_0 + 1|$
4. If $|x - x + 0| < \delta$ then $|x - x_0| < 1$ so $|x + x_0 + 1| < 1 + |2x_0 + 1|$.
5. Then $|f(x) - f(x_0)| = |x - x_0||x + x_0 + 1| < |x - x_0|(1 + |2x_0 + 1| < \delta(1 + |2x_0 + 1|) < \epsilon$
6. Therefore $f(x) = x^2 + x$ is continuous on the set of rational numbers.

Exercise: Prove that $f(x) = x^3$ is continuous on the set of rational numbers.

We proceed with a delta-epsilon proof.

1. $|f(x) - f(x_0)| = |x^3 - x_0^3| = |x - x_0||x^2 + xx_0 + x_0^2|$.
2. Furthermore, $|x^2 + xx_0 + x_0^2| = |x(x + x_0) + x_0^2| = |x||x - x_0 + 2x_0| + |x_0^2|$
3. Let $\delta = min(\frac{\epsilon}{|x||1+2x_0|+|x_0^2|}, 1)$
4. Then $|x||x - x_0 + 2x_0| + |x_0^2| < |x||1 + 2x_0| + |x_0^2|$.
5. Therefore $|x - x_0||x^2 + xx_0 + x_0^2| < (|x||1 + 2x_0| + |x_0^2|)\delta < \epsilon$
6. Thus $f(x) = x^3$ is continuous on the set of rational numbers.

7.2 Undesirable Behavior in $\mathbb{Q}$

Unfortunately, when working with functions that map $\mathbb{Q} \to \mathbb{Q}$, the functions do not always exhibit behavior that is desirable for real analysis. For example, the intermediate value theorem from calculus does not always hold true for the rationals, since continuous functions on the rational numbers do not necessarily take on all intermediate values in an interval. Furthermore, such functions do not necessarily take on maximum or minimum values on closed rational intervals either.

Exercise: For the function $f(x) = x^2 - 2$ for $x \in \mathbb{Q}$, show that $f(x) \in \mathbb{Q} \forall x \in \mathbb{Q}$ and $f(1) < 0$ and $f(2) > 0$. Then show that $f(x)$ is continuous, and that there exists no $x \in \mathbb{Q}$ for which $1 \leq x \leq 2$ and $f(x) = 0$.

First, we show $f(x) \in \mathbb{Q} \forall x \in \mathbb{Q}$ and $f(1) < 0$ and $f(2) > 0$.

1. $f(x) = x * x - 2 = \frac{a*a}{b*b} - \frac{2(b*b)}{b*b}$ with $a, b \in \mathbb{Z}$ by the definition of a rational number.
2. Therefore $f(x) = \frac{a^2 - 2b^2}{b^2}$
3. The value of this expression is rational, because $a^2 - 2b^2 \in \mathbb{Z}$ and $b^2 \in \mathbb{Z}$, therefore $f(x)$ is rational.
4. $f(1) = 1^2 - 2 = -1 < 0$
5. $f(2) = 2^2 - 2 = 2 > 0$

Then, we show $f(x)$ is continuous. It is known that $|f(x) - f(x_0| = |x^2 - 2 - x_0^2 + 2| = |x^2 - x_0^2| = |x - x_0||x - x_0|$ which was already proven to be continuous in the previous exercise for $f(x) = x^2$. Therefore $x^2 - 2$ is continuous.

Finally, if $f(x) = x^2 - 2 = 0$, then $x^2 = 2$. In our previous investigations, we have already proven that $\sqrt{2}$ is irrational, but we have also proven that $f(x)$ only takes on rational values. Therefore, there exists no rational value where $f(x) = 0$. Since $f(x)$ crosses the x-axis in the interval $1 \leq x \leq 2$ due to the fact that $f(1) = 1^2 - 2 = -1 < 0$, $f(2) = 2^2 - 2 = 2 > 0$, and $f(x)$ is continuous and only takes on rational values, we have demonstrated that continuous functions in $\mathbb{Q}$ do not necessarily take on all intermediate values in an interval.

Exercise: For the function $f(x) = \frac{1}{x^2 - 2}$ for $x \in \mathbb{Q}$, show that $f(x) \in \mathbb{Q} \forall x \in \mathbb{Q}$. Then show that $f(x)$ is continuous, and that the function takes on no maximum or minimum values for $x \in \mathbb{Q}$.

First, we demonstrate that $f(x) \in \mathbb{Q}$.

1. If $x \in \mathbb{Q}$ then $x = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $GCD(a, b) = 1$.
2. Then, $f(x) = \frac{1}{(\frac{a}{b})^2 - 2} = \frac{1}{\frac{a^2 - 2b^2}{b^2}} = \frac{b^2}{a^2 - 2b^2}$. Since $b^2 \in \mathbb{Z}$ and $a^2 - 2b^2 \in \mathbb{Z}$, $f(x) \in \mathbb{Q}$.

Next, we show that $f(x)$ is continuous using a similar delta-epsilon argument to those we have been using previously.

1. We know that if $f(x)$ is continuous then $\frac{1}{f(x)}$ is continuous provided $f(x) \neq 0$.
   (a) $|\frac{1}{y} - \frac{1}{y_0}| = |\frac{y-y_0}{yy_0}| = |\frac{y-y_0}{(yy_0-y_0^2+y_0^2)}| = |\frac{y-y_0}{(yy_0-y_0^2+y_0^2)}| = |\frac{y-y_0}{(y_0(y-y_0)+y_0^2)}|$
   (b) Let $\delta = min(\epsilon(y_0^2 - y_0), 1)$. Thus $|y - y_0| < 1 \implies -1 < y - y_0 < 1$. Therefore $y - y_0 > -1$
   (c) Then we can write $|\frac{y-y_0}{(y_0(y-y_0)+y_0^2)}| < |\frac{y-y_0}{(y_0^2-y_0)}| < |\frac{\epsilon(y_0^2-y_0)}{(y_0^2-y_0)}| = \epsilon$ since $x - x_0 < \delta$.
   (d) Therefore, $\frac{1}{y}$ is continuous.
2. Since $f(x)$ is defined only on the rationals, then $f(x) \neq 0$ because that would require $x = \sqrt{2}$ and we already know $\sqrt{2}$ is irrational.
3. We know $x^2 - 2$ is continuous because $|f(x) - f(x_0)| = |x^2 - 2 - x_0^2 + 2| = |x^2 - x_0^2|$ which is identical to the continuity proof of x^2 and we know x^2 is continuous.
4. Therefore $f(x)$ is continuous.

Now, to show that $f(x)$ takes on no maximum or minimum values, we can prove that the range of $f(x)$ is $(-\infty, \infty)$. If we observe a graph of the function in Figure 1, we see that the function has an asymptote at $x = \sqrt{2}$. As $x^2 \to 2$ from the left, $f(x) \to -\infty$, and s $x^2 \to 2$ from the right, $f(x) \to \infty$.

To prove this, we can use a proof by contradiction. Assume $f(x)$ does have a minimum, and let x_1 be the x-value which minimizes $f(x)$ on the interval $1 < x < 2$. Then, we define as $x_2 = 1 + \frac{x_1^2}{2}$. If $x_1 < 2$ then $f(x_2) = \frac{1}{x_1^4+x_1^2-1} < \frac{1}{x_1^2-2}$ when $x > 1$. However, this violates our assumption that x_1 is the minimum. Thus we demonstrate that there is always a smaller value that can be chosen for the x at which $f(x)$ attain its minimum. The same argument can be used to prove the maximum, but when $x > 2$ is approaching from the right.

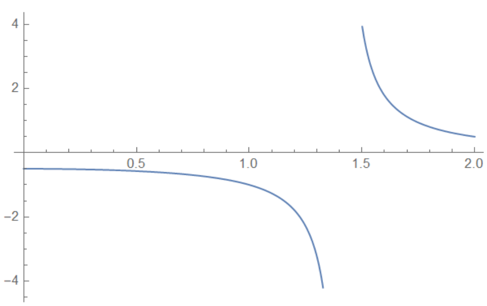


Figure 1: Graph of $f(x) = \frac{1}{x^2-2}$

8 Further Investigation into Limits

8.1 Delta-Epsilon Arguments

Consider a function that maps some subset of $\mathbb{Q}$ to $\mathbb{Q}$.

Then, for a fixed $a \in \mathbb{Q}$, we can define the limit of $f(x)$ as $x \to a$ as L, which is written as $\lim_{x \to a} f(x) = L$, if $\forall \epsilon > 0 \in \mathbb{Q}, \exists \delta > 0 \in \mathbb{Q}$ such that $|x-a| < delta \implies |f(x) - L| < \epsilon$. The function does not need to be defined at a to approach some limit as it approaches a. Since the limit is defined in such a way, we can prove limits of functions using delta-epsilon arguments, similar to continuity.

An example of a proof of a limit is as follows:

Exercise: Prove that $\lim_{x \to 2} \dfrac{x^2 - 4}{x^2 - 3x + 2} = 4$.

Proceed with a delta-epsilon argument.

1. We must prove that $\exists \delta < 0$ such that $|x - 2| < \delta \implies |f(x) - 4| < \epsilon$.
2. Through algebraic it is deduced that $|f(x) - 4| = |\frac{(x-2)(x+2)}{(x-2)(x+1)} - 4| = |\frac{(x-2)+4)}{(x-2)+1} - 4|$
3. Let $\delta = min(\frac{1}{2}, \frac{\epsilon}{2} - 2)$
4. $|x - 2| < \frac{1}{2} \implies -\frac{1}{2} < x - 2 < \frac{1}{2} \implies \frac{1}{2} < x - 2 + 1 < \frac{3}{2}$.
5. Therefore $|\frac{(x-2)+4}{(x-2)+1} - 4| < |2(x-2) + 4| < \epsilon$
6. As such, $\lim_{x \to 2} \dfrac{x^2 - 4}{x^2 - 3x + 2} = 4$

8.2 Problem-Solving with Limits

Next, I include an in-depth description of how I proceeded to complete another example of a limit proof. In class, we continued to discuss problem-solving techniques for proof-writing. Here, I have written down my train of thought and the steps that I took to solve the problem and develop the proof. This description includes errors that I made in the exercise, and the process I took to correct them.

Exercise: Prove that $\lim_{x \to 8} \dfrac{x + 4}{x^2 - 10x + 10} = -2$.

In order to prove the definition of a limit with sufficiently rigorous reasoning, we must used a delta-epsilon argument. This argument is structured as follows: Show that $|x - a| < \delta \implies |f(x) - L| < \epsilon$ for $\delta > 0$ for all $\epsilon > 0$ with δ, ϵ in the set of numbers that you are using for the problem.

In this case we must prove $|x-8| < \delta \implies |\frac{x+4}{x^2-10x+10} - (-2)| < \epsilon$. I'll assume we are operating in the real numbers even though we technically haven't got there yet in class.

So, the only pieces of information I have available are $|x-8| < \delta$ and $|\frac{x+4}{x^2-10x+10} - (-2)| < \epsilon$.

After reading the proofs of continuity of rational functions from the previous section, and observing that these are essentially the same type of argument, I can use a similar strategy to solve the problem. So, the goal is to algebraically manipulate $|\frac{x+4}{x^2-10x+10} - (-2)|$ to allow us to substitute in the inequality $|x-8| < \delta$ to bound the expression below ϵ. By making it clear exactly how δ will be substituted, we can strategically choose a delta that will essentially cancel all of the terms and be equal to epsilon.

So, let's begin with the algebraic manipulation. The goal is to extract $(x-8)$ from $|\frac{x+4}{x^2-10x+10} - (-2)|$.

$$|\frac{x+4}{x^2-10x+10} - (-2)| =$$
$$|\frac{(x-8)+12}{x^2-10x+25-15} + 2| =$$
$$|\frac{(x-8)+12}{(x-5)(x-5)-15} + 2| =$$
$$|\frac{(x-8)+12}{((x-8)+3)((x-8)+3)-15} + 2|$$

So now we have $|f(x) - L|$ written in terms of $|x-a|$. Next, I want to pick a delta such that the expression reduces to ϵ. Previous proofs have used a "min" function to great effect, so I'll start with that. Let $\delta = \min(\text{some constant, some expression with } \epsilon)$

I'll start with the constant, which can be used to eliminate all but one of the $(x-8)$ terms in the expression; I want to leave one term to substitute in the expression with the epsilon. I see $(x-8)+3$ in the denominator, which should be eliminated (since its a denominator). I now realize that in order to bound this function and prevent it from growing to infinity, the denominator needs to be greater than 0, so a simple solution is $(x-8)+3 >= 4$ or $x-8 >= 1$. Thus I need to figure out how to put a lower bound on $x-8$ to solve the problem.

Unfortunately I realized that this is not possible because our expression is $|x-8| < \delta$, or $-\delta < x-8 < \delta$ so we cannot bound the expression $|x-8|$ from below at a positive number. The only way to do that is to start with a different expression and algebraically manipulate it. So I will decide to go back to working with $|\frac{(x-8)+12}{(x-5)(x-5)-15} + 2|$ instead of $|\frac{(x-8)+12}{((x-8)+3)((x-8)+3)-15} + 2|$ I observe that I can algebraically manipulate $x-5$ into $x-8$ by adding 3, so if I want the final lower bound to be 4, the lower bound on $x-8$ needs to be positive anyways.

Or does it? Here is where I got stuck for awhile, but I experimented with different possible solutions to trying to bound the denominator, reaching several dead ends. After a large amount of fiddling about with algebra and experimenting with various bounds, I realize that, I can set an upper bound and ensure that the denominator is always negative to prevent it from going towards 0. If I let $|x-8| < \frac{1}{2}$ then $-\frac{1}{2} < x-8 < \frac{1}{2} \implies \frac{5}{2} < x-5 < \frac{7}{2}$. The square of both of these values is less than 15, so we know that this bound serves to prevent large values of $|f(x)-8|$. Let the constant in the expression for δ by $\frac{1}{2}$. Then we find that $((x-8)+3)((x-8)+3) < \frac{49}{4}$, and further algebra on the denominator yields the inequality $\frac{11}{4} < |(x+8-3)^2 - 15| < \frac{35}{4}$.

Therefore $|\frac{(x-8)+12}{((x-8)+3)((x-8)+3)-15} + 2| < |\frac{4(x-8)+70)}{11}|$

Now we need to find an expression containing ϵ that can be substituted for $(x-8)$ to make this expression equal to epsilon.

Solve algebraically:

$$\begin{aligned} |\frac{4(x-8)+70)}{11}| = \epsilon \implies \\ 4|x-8| = 11(\epsilon) - 70 \implies \\ |x-8| = \frac{11\epsilon - 70}{4} \end{aligned}$$

So we find $\delta = min(\frac{1}{2}, \frac{11\epsilon-70}{4})$.

Then we can complete the delta-epsilon argument by stating that the full proof, which is essentially what I have written here but organized into sequential steps, starting with assuming δ to be what we solved for here.

The final proof is written as a list of steps that reflects the refined course of logical reasoning to prove the limit:

1. We know that $|f(x)-2| = |\frac{x+4}{x^2-10x+10} - (-2)| = |\frac{(x-8)+12}{x^2-10x+25-15} + 2| = |\frac{(x-8)+12}{(x-5)(x-5)-15} + 2| = |\frac{(x-8)+12}{((x-8)+3)((x-8)+3)-15} + 2|$
2. Let $\delta = min(\frac{1}{2}, \frac{5(\epsilon-2)-12}{2})$.
3. It follows that $|x-8| < \frac{1}{2} \implies -\frac{1}{2} < x-8 < \frac{1}{2} \implies \frac{5}{2} < x-8+3 < \frac{7}{2} \implies -\frac{35}{4} < (x+8-3)^2 - 15 < -\frac{11}{4} \implies \frac{11}{4} < |(x+8-3)^2 - 15| < \frac{35}{4}$
4. Therefore $|\frac{(x-8)+12}{(x-5)(x-5)-15} + 2| < |\frac{(x-8)+12}{\frac{11}{4}} + 2| = |\frac{4(x-8)+70}{11}| < \epsilon$
5. Thus $\lim_{x\to 8} \frac{x+4}{x^2-10x+10} = -2$

9 Cauchy Sequences

A Cauchy sequence is a sequence $x : \mathbb{N} \to \mathbb{Q}$ such that $\forall\epsilon > 0, \exists N \in \mathbb{N}$ such that $|x(n+p) - x(n) < \epsilon \forall n > N$ and $\forall p \geq 1$¿ In layman's terms, the difference between a term and another term that comes after the first is arbitrarily small if the sequence goes out far enough.

One of the useful aspects of a Cauchy sequence is that it provides a way for use to concretely define the set of all real numbers $\mathbb{R}$, so we do not have to rely upon all of the assumptions from Section 6. A real number is simply the limit of a Cauchy sequence of rational numbers; a Cauchy sequence that converges to some limit does not necessarily have any other way to describe the number to which it converges, so we call this number "real."

Exercise: Prove that the sequence $x(1) = 1, x(n) = 1 + \frac{1}{1+x(n)}$ is a Cauchy sequence.

First, we will prove the case for $n + 1$. Define $\epsilon(n) = |x(n+1) - x(n) = |1 + \frac{1}{1+x(n)}| = |\frac{2-x(n)^2}{x(n)+1}$ through algebraic manipulation. Then it is known that

$$\begin{aligned}
\epsilon(n+1) &= |1 + \frac{1}{1+x(n+1)} - x(n+1)| \\
&= |1 + \frac{1}{1+1+\frac{1}{1+x(n)}} - 1 - \frac{1}{1+x(n)}| \\
&= |\frac{1+x(n)}{3+2x(n)} - \frac{1}{1+x(n)} \\
&= |\frac{x(n)^2 - 2}{(3+2x(n))(1+x(n))}| \\
&= |\frac{2-x(n)^2}{x(n+1)}||\frac{1}{3+2x(n)}|
\end{aligned}$$

We know that $x(1) = 1$ and the sequence is increasing therefore $\frac{1}{3+2x(n)} < \frac{1}{5}$. As such, $\epsilon(n+1) < \frac{\epsilon(n)}{5} = \frac{1}{5^n}$

Then, we can use the telescopic sum technique to write

$$|x(n+p) - x(n)| = |x(n+p) - x(n+p-1)| + |x(n+p-1) - x(n+p-2)| + \\ ... + |x(n+1) - x(n)|$$

$$< \frac{1}{5^{n+p}} + \frac{1}{5^{n+p-1}} + ... + \frac{1}{5^n}$$

$$= \frac{1}{5^n}\left(\frac{1}{5^p + \frac{1}{5^{p-1}} + ... + 1}\right)$$

$$< \frac{1}{5^n} * 5 = \frac{1}{5^{n-1}}$$

So, given $\epsilon > 0$ we take N large enough such that $\frac{1}{5^{N-1}} < \epsilon$ and this ensures that $|x(n+p) - x(n)| < \epsilon$ for all $n > N$ and $p \geq 1$, making x a Cauchy sequence.

Going further, we can also demonstrate that if a sequence is a Cauchy sequence, then that sequence is bounded, meaning that there exists $c > 0$ such that $|x(n)| \geq C$ for all $n \geq 1$.

One way to do this is a proof by contradiction. Assume that some Cauchy sequence is unbounded. Then, there does not exist $c > 0$ such that $|x(n)| \leq$ for all $n \geq 1$. Therefore, $\lim\limits_{n\to\infty} x(n) = \infty$; the sequence diverges. If this is true then $|x(n+p) - x(n)|$ cannot be less than some arbitrary ϵ since $x(n+p)$ can be infinitely large. This means that the initial assumption that $x(n)$ is a Cauchy sequence is violated and therefore a Cauchy sequence must be bounded. Here, I provide two proofs related to Cauchy sequences.

Exercise: Prove that the sequence $x(1) = 1$, $x(n) = x(n-1) + \frac{1}{1+n}$ is not a Cauchy sequence.

For this exercise, we are proving that there exists an $\epsilon > 0$ such that $|x(n+p) - x(n)| \geq \epsilon$ for all $N \in \mathbb{N}, n > N, p \geq 1$.

1. We know that $x(n)$ is the harmonic series, so we can write $|x(n+p) - x(n)| = \sum_{k=1}^{n+p} \frac{1}{n} - \sum_{k=1}^{n} \frac{1}{n} = \sum_{k=n+1}^{n+p} \frac{1}{n}$
2. We know $n > 1$, so if $\epsilon = 1$, then $|x(n+p) - x(n)| \geq \epsilon$ because the first term of the series is equal to 1.
3. Therefore there exists an $\epsilon > 0$ such that $|x(n+p) - x(n)| \geq \epsilon$ for all $N \in \mathbb{N}$.

Thus $x(n)$ is not a Cauchy sequence.

Exercise: Prove that if x is a convergent sequence, then it is a Cauchy sequence.

1. If x converges then $\lim_{n\to\infty} x(n) = L$.
2. If so, then $x(n+p) < L$
3. Therefore $\epsilon(n) = |x(n+p) - x(n)| < |L - x(n)|$
4. This means that $|x(n+p) - x(n)| < |L - x(n) < \epsilon$
5. However $\lim_{n\to\infty} x(n) = L$, so we can take any N such that $|L - x(n)| < \epsilon$.

Therefore $x(n)$ is a Cauchy sequence.

10 Sequences of Functions

This week we discussed how a sequence of functions can converge to another function. For example, through a numerical investigation, we demonstrated that $f_1(x) = x$, $f_n(x) = f_{n-1}(x) + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$ appears to converge to the sine function as n gets very large. There are two ways that a sequence of functions can converge: uniform convergence and pointwise convergence.

In pointwise convergence, $f(x) = \lim_{n\to\infty} f_n(x)$ implies that f_n defined on the real numbers converges to f on a sub-interval in the real numbers.

Uniform convergence is similar, but it involves looking at all values of x. In uniform convergence, N depends only on ϵ, not on x like it can in pointwise convergence. If a sequence converges uniformly, it also converges pointwise.

A sequence $f_n(x)$ converges uniformly to f if $\forall \epsilon > 0, \exists N \in \mathbb{N}, |f_n(x) - f(x)| < \epsilon, \forall x \in S$.

The next exercise provides an example of a potential problem that can be explored using this concept: the pointwise convergence of the Taylor series for $\sin x$ (or any trigonometric function).

Exercise: Prove pointwise convergence of the sequence $f_1(x) = x$,
$f_n(x) = f_{n-1}(x) + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$.

To prove pointwise convergence, we must prove that $\lim_{n\to\infty} f_n(x) = \sin x$.

To do this, we prove that $\forall \epsilon > 0, \exists N \in \mathbb{N} > 0$ such that $|f_n(x) - \sin x| < \epsilon, \forall n \geq N$.

If we let $|\epsilon(n)| = |f_n(x) - \sin x|$ then $|\epsilon(n+1)| = |\epsilon(n) + (-1)^n \frac{x^{2n+1}}{(2n+1)!}|$.

After further investigation, I found that the solution to this exercise was considerably out of my mathematical wheelhouse, considering it requires some more mathematical machinery that has not yet been covered in this course yet. Although I haven't yet found a proper solution to this exercise, there are variety of ways that I can naively theorize about potential solutions. We know that each iteration seeks to correct the error from the previous. However, if x is too large compared to n, the iteration will overcorrect. At a certain point, overcorrection will not occur and the each iteration of the sequence will yield smaller and smaller error. In this case we want $|\epsilon(n)| < |\frac{x^{2n+1}}{(2n+1)!}| < |2\epsilon(n)|$. Unfortunately, in this case we do not know the previous error.

11 Defining the Real Numbers Using Dedekind Cuts

There are many ways to construct the set of real numbers $\mathbb{R}$, but one particularly simple method is using Dedekind cuts. In order to prove a more intuitive way to understand real numbers, Richard Dedekind defined the numbers in $\mathbb{R}$ by the way in which they split the set of rational numbers into disjoint sets. A number $r \in \mathbb{R}$ can split $\mathbb{Q}$ into two sets: $L : (\frac{a}{b}) < r$ and $(\frac{a}{b}) > r$. Thus, a real number is simply defined as the set of rational numbers less than itself; aka, L. Here, we know that L is non-empty, bounded above (by r) and has no maximum element, since the rationals are infinite. Furthermore, if $\frac{a}{b} \in L$ and $0 < \frac{c}{d} < \frac{a}{b}$ then $\frac{c}{d} \in L$.

11.1 Addition

We can define the addition of two real numbers L_1 and L_2 as the element-wise summation of the two sets.

Exercise: Demonstrate that if L_1 and L_2 are lower cuts, then so is $L_1 + L_2$.

Assume that addition of the real numbers is, as defined above, the element-wise summation of the two sets representing the two real numbers being added.

Let p be the real number represented by the set L_1, and likewise for some q and L_2. Then, if $r_1 = \frac{a}{b} \in L_1$ and $r_1 = \frac{c}{d} \in L_2$ then $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and since $\frac{a}{b} < p$ and $\frac{c}{d} < q$, we know that $\frac{ad+bc}{bd} < p+q$ for any $a, b, c, d \in \mathbb{Z}$. Therefore, $L_1 + L_2$ is a lower cut.

We can also demonstrate the typical properties of addition that make the set of real numbers a field. One example is shown below.

Exercise: Demonstrate that $L_1 + L_2 = L_2 + L_2$.

$L_1 + L_2$ is represented by the element-wise addition of the two sets. We know that addition on the set of real numbers is commutative. Therefore, for any element in $L_1 + L_2$, the corresponding element of $L_2 + L_1$ is the same. Since the sets have the same elements, this means that $L_1 + L_2 = L_1 + L_2$.

We can also extend the same argument to demonstrate other properties of addition such as commutativity with relative ease.

11.2 Multiplication

Multiplication of two real numbers can be defined similarly; that is, $L_1 * L_2$ is the element-wise product of each element in the two sets.

Exercise: Demonstrate that if L_1 and L_2 are lower cuts, then so is $L_1 * L_2$.

Assume that multiplication of the real numbers is, as defined above, the element-wise product of the two sets representing the two real numbers being multiplied.

Let p be the real number represented by the set L_1, and likewise for some q and L_2. Then, if $r_1 = \frac{a}{b} \in L_1$ and $r_1 = \frac{c}{d} \in L_2$ then $\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$ and since $\frac{a}{b} < p$ and $\frac{c}{d} < q$, we know that $\frac{ac}{bd} < p * q$ for any $a, b, c, d \in \mathbb{Z}$. Therefore, $L_1 * L_2$ is a lower cut.

We can also demonstrate the typical properties of multiplication that make the set of real numbers a field. One example is shown below:

Exercise: Demonstrate that $L_1 * (L_2 + L_3) = L_1 * L_2 + L_1 * L_3$.

The real number addition and multiplication is performed via element-wise operations on the rational numbers contained within the corresponding sets.

For elements $\frac{a}{b} \in L_1, \frac{c}{d} \in L_2, \frac{e}{f} \in L_1$ we observe that $\frac{a}{b} * (\frac{c}{d} + \frac{e}{f}) = \frac{acf+ade}{bdf}$ while $\frac{a}{b} * \frac{c}{d} + \frac{a}{b} * \frac{e}{f} = \frac{ac}{bd} + \frac{ae}{bf} = \frac{acbf+abde}{b^2df} = \frac{acf+ade}{bdf}$. This indicates $\frac{a}{b} *$

$(\frac{c}{d} + \frac{e}{f}) = \frac{a}{b} * \frac{c}{d} + \frac{a}{b} * \frac{e}{f}$. Since this is true for all elements of L_1, L_2, L_3, $L_1 * (L_2 + L_3) = L_1 * L_2 + L_1 * L_3$.

11.3 Ordering

The order of real numbers can be defined through the subset operator. The real numbers represented by the sets L_1 and L_2 can be ordered like so: $L_1 \leq L_2$ if $L_1 \subseteq L_2$.

Exercise: Demonstrate that if L_1 and L_2 are lower cuts, then either $L_1 \leq L_2$ or $L_2 \leq L_1$.

Let p be the real number represented by L_1, and likewise for q and L_2. If $L_1, L_2 \in \mathbb{R}$, then these sets are the sets of all real numbers such that $\frac{a}{b} < p$ and $\frac{c}{d} < q$. Thus, L_1 is the set of all elements $\frac{a}{b} < p$ and L_2 is the set of all elements $\frac{c}{d} < q$. If $p = q$, then $L_1 = L_2$; the sets have the same elements. Otherwise, the sets will have unequal numbers of elements In this case, either $L_1 \subset L_2$ or $L_2 \subset L_1$. Without loss of generality, if $L_1 \subset L_2$, then $L_1 \cap L_2 = \{\frac{c}{d} \in L_2 : \frac{c}{d} > p\}$, since L_2 contains elements greater than p, $q > p$ and $L_2 > L_1$.

11.4 Completeness

A subset $S \subseteq \mathbb{R}^+$ is bounded above if $\exists L_1$ such that $L \leq L_1 \forall L \in S$, making L_1 an upper bound for S. The upper bound that is less than all other upper bounds for S is known as the supremum, or $\sup S$.

Exercise: Prove that if $S \subseteq \mathbb{R}^+$ has a least upper bound then the least upper bound is unique.

Proceed with a proof by contradiction: Assume that S has two unique least upper bounds, $\sup S_1$ and $\sup S_2$. Then, $\sup S_1 \neq \sup S_2$ since if they are equal, this contradicts the initial assumption of uniqueness.

If $\sup S_1 \neq \sup S_2$ then $\sup S_1 > \sup S_2$ or $\sup S_1 < \sup S_2$. Both $\sup S_1 > \sup S_2$ and $\sup S_1 < \sup S_2$ violate the definition that $\sup S$ is the least upper bound of S, since there is an upper bound that is less than the assumed supremum, making either $\sup S_1$ or $\sup S_2$ not the least upper bound. Therefore, the least upper bound must be unique.

Exercise: Prove that if $S \subseteq \mathbb{R}^+$ is bounded then $\sup S$ exists.

Proceed with a proof by contradiction: assume that S is bounded but there is no $\sup S$. If $\sup S$ does not exist then either S is unbounded (violating the initial assumption) or the set of all upper bounds of S has no minimum.

Even if the latter is the case, then there is no upper bound to S. This is because, for any feasible bound that one might take, there is always a bound that is smaller than that, since $\sup S$ does not exist and there is no least upper bound. This requires S to not have an upper bound, which contradicts the initial assumption.

11.5 Computability

By one definition, a real number is considered "computable" if we can define a Boolean function B that meets several criteria. First, $\exists r \in \mathbb{Q}^+$ for which $B(r) = 1$ and another such $\exists r \in \mathbb{Q}^+$ for which $B(r) = 0$. Furthermore, if $B(r) = 1$, then $\exists s > r$ with $B(s) = 1$ and if $B(r) = 1$ and $B(s) = 0$ then $r < s$. If these criteria are met then B is a lower cut function for r

Exercise: Show that if $B : \mathbb{Q}^+ \to \{0, 1\}$ is a lower cut function, then $L := \{r \in \mathbb{Q}^+ : B(r) = 1\}$ is a lower cut.

1. If B is a lower cut function, then there exists $r \in \mathbb{Q}^+$ such that $B(r) = 1$. Therefore, it is known that L must be non-empty.

2. Furthermore, if $B(r) = 1$ and $B(s) = 0$ then $r < s$. Because the rational numbers are ordered, for rational numbers a, b, c, if $a < b$ and $b < c$ then $a < c$. Therefore, all values mapped to 0 by B must be greater than all values mapped to 1 by B. Thus, B maps all values r less than some real number to a value of 1. In more mathematical terms, $\forall r \in L, \exists s \notin L > r$. This means that L is bounded above, and that if $r_1 \in L$ and $r_2 < r_1$, $r_2 \in L$.

3. Finally, we know that $\forall r$ such that $B(r) = 1$, $\exists s > r$ with $B(s) = 1$. This indicates that L has no maximum element, since if it did, there would not exist $s > r$ with $B(s) = 1$ for the maximum element.

   Therefore, L has all of the criteria of a lower cut.

References

[1] https://plato.stanford.edu/entries/set-theory/

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