Sequences of Functions

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1 Sequences of Functions

This week we discussed how a sequence of functions can converge to another function. For example, through a numerical investigation, we demonstrated that $f_1(x) = x$, $f_n(x) = f_{n-1}(x) + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$ appears to converge to the sine function as n gets very large. There are two ways that a sequence of functions can converge: uniform convergence and pointwise convergence.

In pointwise convergence, $f(x) = \lim_{n\to\infty} f_n(x)$ implies that f_n defined on the real numbers converges to f on a sub-interval in the real numbers.

Uniform convergence is similar, but it involves looking at all values of x. In uniform convergence, N depends only on ϵ, not on x like it can in pointwise convergence. If a sequence converges uniformly, it also converges pointwise.

A sequence $f_n(x)$ converges uniformly to f if $\forall\epsilon > 0, \exists N \in \mathbb{N}, |f_n(x) - f(x)| < \epsilon, \forall x \in S$.

In this exercise, I attempt to prove pointwise convergence of the Taylor series for $\sin x$

Exercise: Prove pointwise convergence of the sequence $f_1(x) = x$,
$f_n(x) = f_{n-1}(x) + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$.

To prove pointwise convergence, we must prove that $\lim_{n\to\infty} f_n(x) = \sin x$.

To do this, we prove that $\forall\epsilon > 0, \exists N \in \mathbb{N} > 0$ such that $|f_n(x) - \sin x| < \epsilon, \forall n \geq N$.

If we let $|\epsilon(n)| = |f_n(x) - \sin x|$ then $|\epsilon(n+1)| = |\epsilon(n) + (-1)^n\frac{x^{2n+1}}{(2n+1)!}|$.

Although I haven't yet found a proper solution to this exercise, there are variety of ways to go from here. We know that each iteration seeks to correct the error from the previous. However, if x is too large compared to n, the iteration will overcorrect. At a certain point, overcorrection will not occur and the each iteration of the sequence will yield smaller and smaller error. In this case we want $|\epsilon(n)| < |\frac{x^{2n+1}}{(2n+1)!}| < |2\epsilon(n)|$. However, in this case we do not know the previous error.