Cauchy Sequences and Real Numbers

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1 Cauchy Sequences

A Cauchy sequence is a sequence $x : \mathbb{N} \to \mathbb{Q}$ such that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that $|x(n+p) - x(n) < \epsilon \forall n > N$ and $\forall p \geq 1$¿ In layman's terms, the difference between a term and another term that comes after the first is arbitrarily small if the sequence goes out far enough.

One of the most useful aspects of a Cauchy sequence is that it can be used to define the set of all real numbers $\mathbb{R}$. A real number is simply the limit of a Cauchy sequence of rational numbers; a Cauchy sequence that converges to some limit does not necessarily have any other way to describe the number to which it converges, so we call this number "real."

Exercise: Prove that the sequence $x(1) = 1, x(n) = 1 + \frac{1}{1+x(n)}$ is a Cauchy sequence.

First, we will prove the case for $n+1$. Define $\epsilon(n) = |x(n+1) - x(n) = |1 + \frac{1}{1+x(n)}| = |\frac{2-x(n)^2}{x(n)+1}$ through algebraic manipulation. Then it is known that

$$\begin{aligned}
\epsilon(n+1) &= |1 + \frac{1}{1+x(n+1)} - x(n+1)| \\
&= |1 + \frac{1}{1+1+\frac{1}{1+x(n)}} - 1 - \frac{1}{1+x(n)}| \\
&= |\frac{1+x(n)}{3+2x(n)} - \frac{1}{1+x(n)} \\
&= |\frac{x(n)^2 - 2}{(3+2x(n))(1+x(n))}| \\
&= |\frac{2-x(n)^2}{x(n+1)}||\frac{1}{3+2x(n)}|
\end{aligned}$$

We know that $x(1) = 1$ and the sequence is increasing therefore $\frac{1}{3+2x(n)} < \frac{1}{5}$. As such, $\epsilon(n+1) < \frac{\epsilon(n)}{5} = \frac{1}{5^n}$

Then, we can use the telescopic sum technique to write

$$|x(n+p)-x(n)| = |x(n+p)-x(n+p-1)| + |x(n+p-1)-x(n+p-2)| + \\ ... + |x(n+1)-x(n)| \\ < \frac{1}{5^{n+p}} + \frac{1}{5^{n+p-1}} + ... + \frac{1}{5^n} \\ = \frac{1}{5^n}(\frac{1}{5^p + \frac{1}{5^{p-1}} + ... + 1}) \\ < \frac{1}{5^n} * 5 = \frac{1}{5^{n-1}}$$

So, given $\epsilon > 0$ we take N large enough such that $\frac{1}{5^{N-1}} < \epsilon$ and this ensures that $|x(n+p)-x(n)| < \epsilon$ for all $n > N$ and $p \geq 1$, making x a Cauchy sequence.

Going further, we can also demonstrate that if a sequence is a Cauchy sequence, then that sequence is bounded, meaning that there exists $c > 0$ such that $|x(n)| \geq C$ for all $n \geq 1$.

One way to do this is a proof by contradiction. Assume that some Cauchy sequence is unbounded. Then, there does not exist $c > 0$ such that $|x(n)| \leq$ for all $n \geq 1$. Therefore, $\lim\limits_{n\to\infty} x(n) = \infty$; the sequence diverges. If this is true then $|x(n+p)-x(n)|$ cannot be less than some arbitrary ϵ since $x(n+p)$ can be infinitely large. This means that the initial assumption that $x(n)$ is a Cauchy sequence is violated and therefore a Cauchy sequence must be bounded.

Exercise: Prove that the sequence $x(1) = 1$, $x(n) = x(n-1) + \frac{1}{1+n}$ is not a Cauchy sequence.

For this exercise, we are proving that there exists an $\epsilon > 0$ such that $|x(n+p) - x(n)| \geq \epsilon$ for all $N \in \mathbb{N}, n > N, p \geq 1$.

1. We know that $x(n)$ is the harmonic series, so we can write $|x(n+p) - x(n)| = \sum_{k=1}^{n+p} \frac{1}{n} - \sum_{k=1}^{n} \frac{1}{n} = \sum_{k=n+1}^{n+p} \frac{1}{n}$
2. We know $n > 1$, so if $\epsilon = 1$, then $|x(n+p) - x(n)| \geq \epsilon$ because the first term of the series is equal to 1.
3. Therefore there exists an $\epsilon > 0$ such that $|x(n+p) - x(n)| \geq \epsilon$ for all $N \in \mathbb{N}$.

Thus $x(n)$ is not a Cauchy sequence.

Exercise: Prove that if x is a convergent sequence, then it is a Cauchy sequence.

1. If x converges then $\lim_{n\to\infty} x(n) = L$.
2. If so, then $x(n+p) < L$
3. Therefore $\epsilon(n) = |x(n+p) - x(n)| < |L - x(n)|$
4. This means that $|x(n+p) - x(n)| < |L - x(n) < \epsilon$

5. However $\lim_{n\to\infty} x(n) = L$, so we can take any N such that $|L - x(n)| < \epsilon$. Therefore $x(n)$ is a Cauchy sequence.